Article

Phi, Fei, Fo, Fum: Effect Sizes for Categorical Data that Use the Chi-Squared Statistic

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**Abstract:** In both theoretical and applied research, it is often of interest to assess the strength of an observed association. Existing guidelines also frequently recommend going beyond null-hypothesis significance testing and to report effect sizes and their confidence intervals. As such, measures of effect sizes are increasingly reported, valued, and understood. Beyond their value in shaping the interpretation of the results from a given study, reporting effect sizes is critical for meta-analyses, which rely on their aggregation. We here review the most common effect sizes for analyses of categorical variables that use the χ2 (chi-square) statistic, and introduce a new effect size—פ (Fei, pronounced “fay”). We demonstrate the implementation of these measures and their confidence intervals via the *effectsize* package in the R programming language.

**Keywords:** Effect Sizes; Chi-Squared Test; Phi; Cramer’s V, Fei

**MSC:** 62-01; 62-04; 62-08; 62H17; 62P99

1. Introduction

Over the last two decades, there has been growing concerns about the so-called replication crisis in psychology and other fields [1,2]. As a result, the scientific community has paid increasing attention to the issue of replicability in science, as well as to good research and statistical practices.

In this context, many have highlighted the limitations of null-hypothesis significance testing and called for more modern approaches to statistics. One such recommendation coming for example from the “New Statistics” initiative is to report effect sizes and their corresponding confidence intervals, and to increasingly rely on meta-analyses to increase confidence in those estimations [3].[[1]](#footnote-1) These recommendations are meant to complement (or even replace, according to some) null-hypothesis significance testing and would help transition toward a “cumulative quantitative discipline”.

These so-called “New Statistics” are synergistic because effect sizes are not only useful for interpreting study results in themselves, but also because they are necessary for meta-analyses, which aggregate effect sizes and their confidence intervals to create a summary effect size of its own [4,5].

Unfortunately, popular software applications do not always offer the necessary implementations of the specialized effect sizes necessary for a given research design and their confidence intervals. In this paper, we want to focus on effect sizes for categorical data that are probably less well known than popular effect sizes like Cohen’s *d* or Pearson’s *r* [6,7]. For categorical data, *d* and *r* are inappropriate measures of an effect size. Cohen’s *d* refers to the standardized difference between the means of two populations, while Pearson’s correlation coefficient *r* measures linear correlations. Hence, both measures refer to continuous data, not categorical.

To compare categorical data, for instance, where associations can be presented as contingency tables, several effect size metrics are available. Common effect sizes for 2-by-2 tables are odds ratios (OR), risk ratios (RR) or the *phi* () coefficient. While *phi* can be interpreted similarly to a correlation coefficient, OR and RR are harder to interpret as they are not bounded between zero and one. Furthermore, RR is not symmetrical [8]. The effect size can change when columns and rows are exchanged. For tables with larger dimensions than 2-by-2, other effect sizes (like Cramér’s *V*) are available that share the property of *phi* of being able to be interpreted like a correlation coefficient and which are discussed later.

The *observed* distribution of categorical data—usually measured as multinomial variables—can also be compared to an *expected* distribution. Again, effect sizes to measure the strength of such associations show some limitations regarding the ease of interpretation. What is missing here is an effect size whose metric is comparable to those for contingency tables.

This paper aims to review the most commonly used effect sizes for analyses of categorical variables that use the χ2 (chi-square) test statistic, and introduce a new effect size, פ (Fei, pronounced “fay”), which closes the gap of a missing effect size measure in a correlation like metric that is appropriate for categorical data.

Importantly, we offer researchers an applied walkthrough on how to use these effect sizes in practice thanks to the *effectsize* package in the R programming language, which implements these measures and their confidence intervals [9,10]. The presented *effectsize* package closes another gap related to the aforementioned effect sizes, because the uncertainty of such measures—expressed by their confidence intervals—is often not included in the output of statistical software. We cover in turn tests of independence (φ/phi, Cramér’s *V*) and tests of goodness of fit (Cohen’s *w*, Tschuprow’s *T* and a new proposed effect size, פ/Fei).

2. Effect sizes for tests of independence

The χ2 test of independence between two categorical variables examines if the frequency distribution of one of the variables is dependent on the other. That is, are the two variables correlated such that, for example, members of group 1 on variable X are more likely to be members of group A on variable Y, rather than evenly spread across Y variable groups A and B. Formally, the test examines how likely the observed conditional frequencies (cell frequencies) are under the null hypotheses of independence. This is done by examining the degree the observed cell frequencies deviate from the frequencies that would be expected if the variables were indeed independent. The test statistic for these tests is the χ2, which is computed as:

where are the *observed* frequencies and are the frequencies *expected* under independence, and and are the number of rows and columns, respectively, of the contingency table.

Instead of the deviations between the observed and expected frequencies, we can write χ2 in terms of observed and expected cell *probabilities* and the total sample size (since):

where are the *observed* cell probabilities and the probabilities *expected* under independence.

Table 1 gives a short example in R to demonstrate whether the probability of survival is dependent on the sex of the passenger aboard the Titanic. The null hypothesis tested here is that the probability of survival is independent of the passenger’s sex.

**Table 1.** χ2 test of survival of Titanic passengers by sex, Titanic dataset from R

|  |  |  |
| --- | --- | --- |
| **Sex** | **Survived** | **Died** |
| Male | 367 | 1364 |
| Female | 344 | 126 |

χ2 = 456.9, *df* = 1, *p* < .001\*\*\*

The performed χ2-test is statistically significant. Thus, we can reject the hypothesis of independence. However, the output includes no effect size, and we cannot conclude the strength of the association between sex and survival.

2.1 Phi

For a 2-by-2 contingency table analysis, as the one used above, the (*phi*) coefficient is a correlation-like measure of effect size indicating the strength of association between the two binary variables. One possibility to compute this effect size is to re-code the binary variables as dummy (“0” and “1”) variables, and compute the Pearson correlation between them [11]:

Another way to compute is by using the statistic:

This value ranges between 0 (no association) and 1 (complete dependence), and its values can be interpreted the same as Person’s correlation coefficient. Table 2 shows the correlation coefficient and the effect size for the data shown in table 1.

**Table 2.** Correlation and effect size (*phi*) for the survival of Titanic passengers by sex, Titanic dataset from R

|  |  |  |  |
| --- | --- | --- | --- |
| **Variable 1** | **Variable 2** | ***r* (95% CI)** | **(95% CI)** |
| Sex (male/female) | Survival (survived/died) | -.46 (-.49, -.42) | .46 (.42, 1.00) |

Note that cannot be negative, so will take the *absolute* value of Pearson’s correlation coefficient. Also note thatthe *effectsize* package gives a *one-sided* confidence interval by default, to match the positive direction of the associated test at (that the association is *larger* than zero at a 95% confidence level).

2.2 Cramér’s V (and Tschuprow’s T)

When the contingency table is larger than 2-by-2, using can produce values larger than 1, and so loses its interpretability as a correlation-like effect size. Cramér showed that while for 2-by-2 the maximal possible value of is , for larger tables the maximal possible value for is [12]. Therefore, he suggested the effect size (also sometimes known as Cramér’s phi and denoted as ):

where is 1 when the columns are completely dependent on the rows, or the rows are completely dependent on the columns (and 0 when rows and columns are completely independent).

**Table 3.** Effect size Cramér’s *V* for the survival of Titanic passengers by class/position, Titanic dataset from R

|  |  |  |
| --- | --- | --- |
| **Class/Position** | **Survived** | **Died** |
| **1st** | 203 | 122 |
| **2nd** | 118 | 167 |
| **3rd** | 178 | 528 |
| **Crew** | 212 | 673 |

Cramér’s *V* = .29, 95% CI = .26, 1.00

Tschuprow devised an alternative value, at

which is 1 only when the columns are completely dependent on the rows *and* the rows are completely dependent on the columns, which is only possible when the contingency table is a square [13].

For example, in Table 4, each row is dependent on the column value; that is, if we know if the food is a soy, milk, or meat product, we also know whether the food is vegan or not. However, the columns are *not* fully dependent on the rows: knowing the food is vegan tells us the food is soy based, however, knowing it is not vegan does not allow us to classify the food—it can be either a milk product or a meat product.

Accordingly, as can be seen in Table 4, Cramer’s *V* will be 1, but Tschuprow’s *T* will not be:

**Table 4.** Cramér’s *V* and Tschuprow’s *T* for food classes, example dataset from R

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Type** | **Product** | | | **Cramér’s *V* (95% CI)** | **Tschuprow’s *T* (95% CI)** |
| **Soy** | **Milk** | **Meat** |
| Vegan | 47 | 0 | 0 | 1.00 (.81, 1.00) | .84 (.68, 1.00) |
| Not-Vegan | 0 | 12 | 12 |

We can generalize , , and to: . That is, they express the square root of a proportion of the sample- to the maximum possible given the study design.

These coefficients can also be used for confusion matrices, which are 2-by-2 contingency tables used to assess machine learning algorithms’ classification abilities, comparing true outcome classes with the model-predicted outcome class. A popular metric is the Matthews correlation coefficient (MCC) for binary classifiers, which is often presented in terms of true and false positives and negatives, but is nothing more than [14].

3. Effect sizes for goodness of fit tests

These tests compare an observed distribution of a multinomial variable to an expected distribution, using the same statistic. Here, in addition, we can compute an effect size as ; all we need to find is .

3.1 Cohen’s w

Cohen defined an effect size—*w*—for the goodness of fit test [7]:

Thus, .

Unfortunately, *w* has an upper bound of 1 *only* when the variable is binomial (has two categories) and the expected distribution is uniform (). When the distribution is non-uniform or if there are more than two classes, then , and so *w* can be larger than 1 [15,16]. Examples are shown in Table 5.

**Table 5.** Effect size Cohen’s *w* for variables with different numbers of categories and distributions

|  |  |  |
| --- | --- | --- |
| **Observed counts** | **Expected proportion** | **Cohen’s *w* (95% CI)** |
| 90 / 10 | .5 / .5 | 0.80 (0.61, 1.00) |
| 90 / 10 | .35 / .65 | 1.15 (0.99, 1.36) |
| 5 / 10 / 80 / 5 | .25 / .25 / .25 / .25 | 1.27 (1.10, 1.73) |

Although Cohen suggested that *w* can also be used for such designs, we believe that this hinders the interpretation of *w* since it can be arbitrarily large [7].

3.2 Fei

We present here a new effect size, פ (Fei, pronounced “fay”), which normalizes goodness-of-fit by the proper for non-uniform and/or multinomial variables.

The largest deviation from the expected probability distribution would occur when all observations are in the cell with the smallest expected probability. That is:

We can find for each of these values:

Therefore,

And so,

Finally, an effect size can be derived as:

We call this effect size פ (Fei),[[2]](#footnote-2) which represents the voiceless bilabial fricative in the Hebrew language, keeping in line with (which in modern Greek marks the same sound) and (which in English marks a voiced bilabial fricative; being derived from the letter V in the modern Latin alphabet). פ will be 0 when the observed distribution perfectly matches the one expected (under the null hypothesis) and will be 1 when the sample contains *only* one class of observations—the one with the smallest expected probability (under the null hypothesis). That is, פ is 1 (its maximal value) only when we observe only the least expected class. When there are only two cells with uniform expected probabilities (50%), the expression reduces to and so פ . Table 6 shows the effect size Fei for the same vectors and distributions as seen for Cohen’s *w* in Table 5. As can be seen, unlike Cohen’s *w*, all effect size values of Fei (and their confidence intervals) are within the range from 0 to 1.

**Table 6.** Effect size Fei for variables with different numbers of categories and distributions

|  |  |  |
| --- | --- | --- |
| **Observed counts** | **Expected proportion** | **Fei (95% CI)** |
| 90 / 10 | .5 / .5 | .80 (.64, 1.00) |
| 90 / 10 | .35 / .65 | .85 (.73, 1.00) |
| 5 / 10 / 80 / 5 | .25 / .25 / .25 / .25 | .73 (.64, 1.00) |

Computation of פ (Fei) can be achieved with the fei() function of the *effectsize* package—fei(c(90, 10), p = c(0.35, 0.65)). This function here computed confidence intervals using the noncentrality parameter method (also called the "pivot method"), by finding values for the noncentrality parameter ("ncp") of a noncentral χ2 distribution that places the observed χ2 test statistic at the desired probability point of the distribution (e.g., *p* = .025 and *p* = .975 for a two-sided 95% confidence interval). The two ncps (for the lower and upper bounds) are then converted back to פ using the same scale-and-root formula as with the sample χ2:

where and are the noncentrality parameters corresponding to the desired tail probabilities, and and are the bounds of the confidence intervals for פ (Fei). (See also next section.)

4. Simulation study of the distributional form of the Fei effect size

In the previous section, we showed some results for the effect size פ (Fei) and its confidence intervals for different distributions of a multinomial variable. Like all effect sizes discussed in this paper, פ follows a scaled non-central distribution: the statistic follows a non-central distribution, and its square root follows a non-central distribution; this random variable is then scaled by a constant that is a function of the sample size and the study design. The noncentrality parameter of the non-central distribution can be found by applying the inverse of the scale on the population effect size. Therefore

where is the inverse of the to פ conversion:

פ is the population effect size, *k* is the number of classes, and is the random variable of possible observed effect sizes in a random sample. This can also be formulated in terms of a non-central distribution:

where is the inverse of the to פ conversion:

To validate our assumptions, we conducted a simulation study, where we simulated data of multinomial distributions for known true effect sizes of 0.1, 0.3 and 0.5, respectively. The datasets contained 500 simulations per effect size, for three different expected probabilities (same as in Table 6) and three different sample sizes of size 50, 100, and 350, resulting in 13,500 simulated data points (500 simulations × 3 effect sizes × 3 expected probabilities × 3 different sample sizes). Figure 1 shows the results from the simulation study.

Figure 1: Comparison of true and simulation-based actual effect size פ (Fei) for different expected proportions, true effect sizes, and sample sizes. Histograms represent the distributions of the sample פ’s from the simulated datasets, and the plotted lines represent density functions for (scaled) non-central distributions for the corresponding effect sizes and sample sizes.



The smallest sample size is more affected by noise and results show more variation (and less continuity) of simulation-based פ (Fei) values around the true effect sizes. For sample sizes *N* = 100 and *N* = 350, פ values closely replicate the true effect sizes and clearly follow a non-central distribution, indicating that Fei, like , *V*, *T*, and *w*, is a scaled χ value.

פ (Fei) following a non-central χ distribution also allows for power calculation. For example, if the null probabilities are [0.35, 0.65] and the alternative probabilities are [0.545, 0.455], the scaling constant is:

and the population effect size is

Therefore, the sample פ will follow the following distribution:

One must then find the *N* that produces the desired power for the significance level that will be used to reject the null. For example, for a significant level of .01 and a power of at least .85, an *N* of at least 78 is required.

The *pwr* package in R provides a function (pwr.chisq.test()) that can be used for power calculated for goodness-of-fit tests. Although the function uses Cohen’s *w* as input/output, פ can easily be converted to Cohen’s *w* (e.g., by using the fei\_to\_w() function from *effectsize*), allowing for the *pwr* function to be used with פ. An example can be found in the accompanying R code.

5. Conclusion

Effect sizes are essential to interpret the magnitude of observed effects, they are frequently required in scientific journals, and they are necessary for a cumulative quantitative science relying on meta-analyses. In this paper, we have covered the mathematics and implementation in R of four different effect sizes for analyses of categorical variables that specifically use the (chi-square) statistic. Furthermore, with our proposal of the effect size פ (Fei), we fill in the missing effect size for all cases of a test.

**Table 7.** Effect size for tests for differently sized contingency tables

|  |  |  |
| --- | --- | --- |
| **Test** | **Table Size** | **Effect size** |
| test for independence | 2-by-2 |  |
| Larger than 2-by-2 | or  (Reduces to when table is 2-by-2) |
| test for goodness-of-fit | 2 classes with uniform null distribution |  |
| More than 2 classes and/or non-uniform null distribution | (Reduces to when there are  2 classes with uniform null dist.) |

Thus, we now have effect sizes to accompany any sized 1-dimensional or 2-dimensional contingency tables, that represent the sample’s relative to the maximally possible , ranging from 0 to 1, that can be easily interpreted on the scale of a correlation coefficient.

**Author Contributions:** M.S.B. conceptualized and developed the Fei effect size and its implementation in *effectsize*, conducted the simulation study, and drafted the paper; D.L. prepared the revised manuscript; all authors contributed to both the writing of the paper and the conception of the software. All authors have read and agreed to the published version of the manuscript.

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1. The title of this paper is an allusion to the rhyme spoken by the giant in the English fairy tale Jack and the Beanstalk (“Fee-fi-fo-fum”). [↑](#footnote-ref-1)
2. The Hebrew character can be inserted into documents via several methods:

   By copying the character from <https://util.unicode.org/UnicodeJsps/character.jsp?a=05E4> or similar webpages.

   In R, by typing the string "\u05e4"

   In LaTeX, by typing \char"05e4 and using a Unicode-compatible compiler, such as XeTeX or LuaLaTeX

   In Microsoft Word, from the Hebrew character of the Symbols window (Insert → Symbol…) or by typing *05e4*, followed by *Alt + X* on the keyboard (Windows only).

   On Windows, using the Character Map application or by holding down *Alt* and typing *+1508* on the numeric keypad.

   On macOS, by enabling the Unicode Hex Input language from System Settings… → Keyboard, then typing *Opt + 05e4*. [↑](#footnote-ref-2)